CONSTRUCTION OF POLYNOMIAL LATTICE RULES OVER $\mathbb{F}_2$ WITH SMALL MEAN SQUARE WEIGHTED $\mathcal{L}_2$ DISCREPANCY*

TAKASHI GODA†

Abstract. The $\mathcal{L}_2$ discrepancy is one of the well-known quantitative measures for the irregularity of distribution of point sets in the unit cube. The weights were introduced by Sloan and Woźniakowski to take into account the relative importance of the discrepancy of lower dimensional projections. As known under the name of quasi-Monte Carlo methods, point sets with small weighted $\mathcal{L}_2$ discrepancy are of use in numerical integration. In this study, we study the component-by-component construction of polynomial lattice rules over the finite field $\mathbb{F}_2$ whose scrambled point sets have small mean square weighted $\mathcal{L}_2$ discrepancy. We prove an upper bound on this discrepancy which converges at almost the best possible rate of $N^{-2+\delta}$ for all $\delta > 0$, where N denotes the number of points. Numerical experiments confirm that the performance of our constructed polynomial lattice rules is comparable or even superior to that of the well-known Sobol' sequences.

Key words. polynomial lattice rules, weighted $\mathcal{L}_2$ discrepancy, numerical integration, randomized quasi-Monte Carlo methods

AMS subject classifications. 11K38, 65D30, 65D32

1. Introduction. In this paper, we study the approximation of an s-dimensional integral over the unite cube $[0,1)^s$

$$I(f) = \int_{[0,1)^s} f(\boldsymbol{x}) d\boldsymbol{x},$$

by averaging with equal weights the function values evaluated at N points

$$Q(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n).$$

Monte Carlo (MC) and quasi-Monte Carlo (QMC) methods choose the point set $P_{N,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ randomly and deterministically, respectively. The aim of QMC methods is to distribute the quadrature points as uniformly as possible so as to yield the smaller integration error. This idea is supported by the general form of various integration error bounds

$$|I(f) - Q(f)| \le V(f) D(P_{N,s}), \tag{1.1}$$

where $V(f)$ is the variation of the integrand f in some sense, which depends only on f, while $D(P_{N,s})$ is the corresponding discrepancy of the point set $P_{N,s}$, which measures the irregularity of distribution of $P_{N,s}$ and depends only on $P_{N,s}$. Thereby, as the smaller $D(P_{N,s})$ is, the smaller integration error we can expect. The most well-known bound is given by the so-called Koksma-Hlawka inequality in which $V(f)$ is the variation of f in the sense of Hardy and Krause and $D(P_{N,s})$ is the star discrepancy of $P_{N,s}$, see for example [17, 20].

Randomization of the QMC point set is helpful to obtain statistical information on the integration error and sometimes even enables us to improve the rate of convergence for numerical integration. There have been several methods introduced for

*The support of Grant-in-Aid for JSPS Fellows No.24-4020 is gratefully acknowledged.

†Graduate School of Information Science and Technology, The University of Tokyo, 2-11-16 Yayoi, Bunkyo-ku, Tokyo 113-0032 (goda@iba.t.u-tokyo.ac.jp).

randomization [5, 18, 26, 36]. Using the linearity of expectation and (1.1), the mean square integration error is upper-bounded by

$$\mathbb{E}\left[|I(f)-Q(f)|^2\right] \le V^2(f)\mathbb{E}\left[D^2(P_{N,s})\right],$$

where the expectation is taken with respect to all the possible randomized point sets of $P_{N,s}$. Hence, the mean square discrepancy becomes a meaningful measure of the irregularity of distribution of $P_{N,s}$ in this setting.

Among others for $D(P_{N,s})$, the $\mathcal{L}_2$ discrepancy is one of the popular measures of the irregularity of distribution of point sets. The relationship between the $\mathcal{L}_2$ discrepancy and numerical integration has been often discussed in the literature, see for example [11, 23, 34, 38, 39]. Sloan and Woźniakowski [34] introduced a concept of the weighted $\mathcal{L}_2$ discrepancy to take the relative importance of the discrepancy of lower dimensional projections into account. It provides a part of the reason why QMC methods are successful even for very large values of s, as often reported in the practical applications to financial problems [3, 22, 29]. This phenomenon is hard to explain by the classical integration error bounds. Hence, construction of point sets with small weighted $\mathcal{L}_2$ discrepancy is of particular interest to QMC practitioners. Especially, in this paper, we focus on constructing randomized QMC point sets with small mean square weighted $\mathcal{L}_2$ discrepancy.

In order to give the definition of the weighted $\mathcal{L}_2$ discrepancy, we introduce some notation first. For a point set $P_{N,s}=\{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$ in the unit cube $[0,1)^s$, the local discrepancy function is defined as

$$\Delta(\boldsymbol{t}) := \frac{A_N([\boldsymbol{0},\boldsymbol{t}),P_{N,s})}{N} - t_1\cdots t_s,$$

where $\boldsymbol{t}=(t_1,\dots,t_s)$ is the vector from $[0,1)^s$, $[\boldsymbol{0},\boldsymbol{t})$ is the axis-parallel box of the form $[0,t_1)\times\cdots\times[0,t_s)$, and $A_N([\boldsymbol{0},\boldsymbol{t}),P_{N,s})$ denotes the number of indices n with $\boldsymbol{x}_n\in[\boldsymbol{0},\boldsymbol{t})$. Let $I_s=\{1,\dots,s\}$ and let γ_u be a non-negative real number for $u\subseteq I_s$. We denote by $|u|$ the cardinality of u and by $\boldsymbol{t}_u$ the vector from $[0,1)^{|u|}$ containing all the components of $\boldsymbol{t}\in[0,1)^s$ whose indices are in u. Further, let $d\boldsymbol{t}_u=\prod_{j\in u}dt_j$ and let $(\boldsymbol{t}_u,\boldsymbol{1})$ denote the vector from $[0,1)^s$ with all the components whose indices are not in u replaced by one. Then, the weighted $\mathcal{L}_2$ discrepancy of the point set $P_{N,s}$ is given by

$$\mathcal{L}_{2,N,\boldsymbol{\gamma}}(P_{N,s}) = \left(\sum_{\emptyset\ne u\subseteq I_s}\gamma_u\int_{[0,1]^u}|\Delta(\boldsymbol{t}_u,\boldsymbol{1})|^2 d\boldsymbol{t}_u\right)^{1/2}.$$

We can recover the classical $\mathcal{L}_2$ discrepancy by choosing $\gamma_{I_s}=1$ and $\gamma_u=0$ for $u\subset I_s$. The following proposition generalizes the well-known formula for the classical $\mathcal{L}_2$ discrepancy introduced by Warnock, see for example [8, 19].

Proposition 1.1. *For any point set $P_{N,s}=\{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and any sequence $\boldsymbol{\gamma}=(\gamma_u)_{u\subseteq I_s}$ of weights, we have*

$$\begin{aligned}&\mathcal{L}^2_{2,N,\boldsymbol{\gamma}}(P_{N,s})\\ &= \sum_{\emptyset\ne u\subseteq I_s}\gamma_u\left[\frac{1}{3^{|u|}}-\frac{2}{N}\sum_{n=0}^{N-1}\prod_{j\in u}\frac{1-x_{n,j}^2}{2}+\frac{1}{N^2}\sum_{n,n'=0}^{N-1}\prod_{j\in u}(1-\max(x_{n,j},x_{n',j}))\right],\end{aligned}$$

where $x_{n,j}$ is the j-th component of the point $\boldsymbol{x}_n$.

There are two prominent construction principles of the QMC point sets: lattice rules [20, 32] and digital (t, m, s)-nets [8, 20]. In this study, we are concerned with polynomial lattice rules which can be categorized into the latter, while its name comes from the analogy with lattice rules. Since first introduced by Niederreiter [21], polynomial lattice rules have been extensively investigated, see for example [8, 16]. In the following, we give the definition of polynomial lattice rules for the case of base 2 because we will only deal with that case.

Let $\mathbb{F}_2 := \{0, 1\}$ be the two element field and represent by $\mathbb{F}_2((x^{-1}))$ the field of formal Laurent series over $\mathbb{F}_2$. Every element of $\mathbb{F}_2((x^{-1}))$ has the form

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

where w is an arbitrary integer and all $t_l \in \mathbb{F}_2$. Further, we denote by $\mathbb{F}_2[x]$ the set of all polynomials over $\mathbb{F}_2$. For a given integer m, we define the map v_m from $\mathbb{F}_2((x^{-1}))$ to the interval $[0, 1)$ by

$$v_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{m} t_l 2^{-l}.$$

We often identify a non-negative integer k whose dyadic expansion is given by $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_a 2^a$ with the polynomial over $\mathbb{F}_2[x]$ as $k(x) = \kappa_0 + \kappa_1 x \cdots + \kappa_a x^a$. For $\boldsymbol{k} = (k_1 \cdots, k_s) \in \mathbb{F}_2[x]^s$ and $\boldsymbol{q} = (q_1 \cdots, q_s) \in \mathbb{F}_2[x]^s$, we define the inner product as

$$\boldsymbol{k} \cdot \boldsymbol{q} = \sum_{j=1}^{s} k_j q_j \in \mathbb{F}_2[x],$$

and we write $q \equiv 0 \pmod{p}$ if p divides q in $\mathbb{F}_2[x]$. Using these notations, the polynomial lattice point set is constructed as follows.

Definition 1.2. *Let $m, s \in \mathbb{N}$. Let $p \in \mathbb{F}_2[x]$ be an irreducible polynomial with $\deg(p) = m$ and let $\boldsymbol{q} = (q_1, \ldots, q_s) \in \mathbb{F}_2[x]^s$. The polynomial lattice point set $P_{2^m,s}(\boldsymbol{q}, p)$ is the point set consisting of the 2^m points*

$$\boldsymbol{x}_n := \left(v_m\left(\frac{n(x)q_1(x)}{p(x)}\right), \ldots, v_m\left(\frac{n(x)q_s(x)}{p(x)}\right)\right) \in [0, 1)^s,$$

for $n \in \mathbb{F}_2[x]$ with $\deg(n) < m$.

In the following, the notation $P_{2^m,s}(\boldsymbol{q}, p)$ implicitly means that $\deg(p) = m$ and the number of components for a vector $\boldsymbol{q}$ is s.

For randomization of the polynomial lattice point set, we consider to apply Owen's scrambling [26, 27, 28]. It proceeds as follows. For $\boldsymbol{x} = (x_1, \cdots, x_s) \in [0, 1)^s$, we denote the dyadic expansion by $x_j = x_{j,1}2^{-1} + x_{j,2}2^{-2} + \cdots$. Let $\boldsymbol{y} = (y_1, \cdots, y_s) \in [0, 1)^s$ be the scrambled point of $\boldsymbol{x}$ whose dyadic expansion is represented by $y_j = y_{j,1}2^{-1} + y_{j,2}2^{-2} + \cdots$. Each coordinate y_j is obtained by applying permutations to each digit of x_j. Here the permutation applied to $x_{j,k}$ depends on $x_{j,l}$ for $1 \le l \le k-1$. In particular, $y_{j,1} = \pi_j(x_{j,1})$, $y_{j,2} = \pi_{j,x_{j,1}}(x_{j,2})$, $y_{j,3} = \pi_{j,x_{j,1},x_{j,2}}(x_{j,3})$, and in general

$$y_{j,k} = \pi_{j,x_{j,1},\cdots,x_{j,k-1}}(x_{j,k}),$$

where $\pi_{j,x_{j,1},\cdots,x_{j,k-1}}$ is a random permutation of $\{0, 1\}$. We choose permutations with different indices mutually independent from each other where each permutation

is chosen with the same probability. Then, as shown in [26, Proposition 2], the scrambled point $\boldsymbol{y}$ is uniformly distributed in $[0,1)^s$.

Our aim here is to find a vector $\boldsymbol{q}$ with p fixed, which yields small mean square weighted $\mathcal{L}_2$ discrepancy. Restricting each $q_j \in \mathbb{F}_2[x]$ such that $\deg(q_j) < m$ and $q_j \neq 0$, the number of candidates for $\boldsymbol{q}$ is $(2^m-1)^s$, which is quite large. The component-by-component (CBC) construction can significantly reduce the computational burden by searching over all the candidates of q_{j+1} while leaving the existing components $(q_1,\dots,q_j)$ unchanged. The CBC construction was first invented for lattice rules by Korobov [13] and re-discovered more recently by Sloan and Reztsov [33]. It also has been applied to polynomial lattice rules. Without requiring exhaustive search, the CBC construction usually finds a good vector $\boldsymbol{q}$ as discussed in many previous studies, see for example [2, 6, 7, 14, 15]. Hence, we employ the CBC construction to find a vector $\boldsymbol{q}$ which gives small mean square weighted $\mathcal{L}_2$ discrepancy.

We end this section with a brief outline of this paper. In the next section, we introduce Walsh functions and their useful properties. They play a central role in the analysis of the mean square weighted $\mathcal{L}_2$ discrepancy. In Section 3, we study the mean square weighted $\mathcal{L}_2$ discrepancy of scrambled polynomial lattice rules. Next, in Section 4, we construct polynomial lattice rules which have small mean square weighted $\mathcal{L}_2$ discrepancy. We prove an upper bound on the root of this discrepancy which converges at a rate of $N^{-1+\delta}$ for all $\delta > 0$, where $N = 2^m$ denotes the number of points. As Roth [31] and Proinov [30] proved that the lower bound on the classical $\mathcal{L}_2$ discrepancy of N points is given by

$$\mathcal{L}_{2,N,\boldsymbol{\gamma}}(P_{N,s}) \geq c_s \frac{(\log N)^{(s-1)/2}}{N}, \tag{1.2}$$

where c_s is a constant dependent only on s, our upper bound is almost best possible in the sense that a rate of N^{-1} cannot be achievable. We further mention about the strong tractability of our construction algorithm. Finally, in Section 5, we show the performance of our constructed polynomial lattice rules and compare with that of the well-known Sobol' sequences.

2. Walsh functions. Walsh functions were first introduced by Walsh [37] and have been extensively studied for example in [4, 10]. We refer to [8, Appendix A] for more information on Walsh functions. In the following, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denotes a set of non-negative integers. We first give the definition of dyadic Walsh functions for the one-dimensional case.

DEFINITION 2.1. *Let $k \in \mathbb{N}_0$ with dyadic expansion $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_a 2^a$. Then, the k-th dyadic Walsh function* $\mathrm{wal}_k : [0,1) \to \{-1,1\}$ *is defined as*

$$\mathrm{wal}_k(x) = (-1)^{x_1\kappa_0 + \cdots + x_{a+1}\kappa_a},$$

for $x \in [0,1)$ with dyadic expansion $x = x_1 2^{-1} + x_2 2^{-2} + \cdots$ (unique in the sense that infinitely many of the x_i must be zero).

This definition can be generalized to higher-dimensional case.

DEFINITION 2.2. *For $s \in \mathbb{N}$, let $\boldsymbol{x} = (x_1, \cdots, x_s) \in [0,1)^s$ and $\boldsymbol{k} = (k_1, \cdots, k_s) \in \mathbb{N}_0^s$. We define* $\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \{-1,1\}$ *by*

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

In the following, the operator $\oplus$ denotes the digitwise addition modulo 2, that is, for $x, y \in [0,1)$ with dyadic representations $x = \sum_{i=1}^{\infty} x_i 2^{-i}$ and $y = \sum_{i=1}^{\infty} y_i 2^{-i}$, $\oplus$ is defined as

$$x \oplus y = \sum_{i=1}^{\infty} z_i 2^{-i},$$

where $z_i = x_i + y_i \pmod 2$. We also define a digitwise addition for non-negative integers based on those dyadic representations. In case of vectors in $[0,1)^s$ or $\mathbb{N}_0^s$, the operator $\oplus$ is carried out componentwise. Further, we call $x \in [0,1)$ a dyadic rational if it can be represented by a finite dyadic expansion. The proposition below summarizes some basic properties that Walsh functions hold.

PROPOSITION 2.3. *We have the following:*

1. *For all $k, l \in \mathbb{N}$ and all $x, y \in [0,1)$ with the restriction that if x, y are not dyadic rationals, then $x \oplus y$ is not allowed to be a dyadic rational, we have*

$$\mathrm{wal}_k(x)\mathrm{wal}_l(x) = \mathrm{wal}_{k\oplus l}(x),\ \ \mathrm{wal}_k(x)\mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus y).$$

2. *We have*

$$\int_0^1 \mathrm{wal}_0(x)dx = 1 \quad \text{and} \quad \int_0^1 \mathrm{wal}_k(x)dx = 0 \quad \text{if } k \in \mathbb{N}.$$

3. *For all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$, we have*

$$\int_{[0,1)^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{x})d\boldsymbol{x} = \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{l}, \\ 0 & \text{otherwise.} \end{cases}$$

4. *For $s \in \mathbb{N}$, the system $\{\mathrm{wal}_{\boldsymbol{k}} :\ \boldsymbol{k} = (k_1 \ldots, k_s) \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $\mathcal{L}_2([0,1]^s)$.*

Furthermore, in order to introduce an important relation between Walsh functions and polynomial lattice rules as described below in Lemma 2.5, we add one more notation and introduce a concept of the so-called *dual net* of a polynomial lattice point set $P_{2^m,s}(\boldsymbol{q}, p)$. For $k \in \mathbb{N}_0$ with dyadic expansion $k = k_0 + k_1 2 + \cdots$, $\mathrm{tr}_m(k)$ gives a polynomial of degree at most m by truncating the associated polynomial $k(x) \in \mathbb{F}_2[x]$ as

$$\mathrm{tr}_m(k) = k_0 + k_1 x + \cdots + k_{m-1}x^{m-1}.$$

For a vector $\boldsymbol{k} = (k_1 \cdots, k_s) \in \mathbb{N}_0^s$, we define $\mathrm{tr}_m(\boldsymbol{k}) = (\mathrm{tr}_m(k_1), \cdots, \mathrm{tr}_m(k_s))$. With this notation, we introduce the following definition of the dual net $D^*_{\boldsymbol{q},p}$.

DEFINITION 2.4. *The dual net for a polynomial lattice point set $P_{2^m,s}(\boldsymbol{q}, p)$ is given by*

$$D^*_{\boldsymbol{q},p} = \{\boldsymbol{k} \in \mathbb{N}_0^s :\ \mathrm{tr}_m(\boldsymbol{k}) \cdot \boldsymbol{q} \equiv 0 \pmod p\}.$$

Then, the following lemma relates the dual net of a polynomial lattice point set to the numerical integration of Walsh functions. It follows immediately from Definition 2.4, [8, Lemma 10.6] and [8, Lemma 4.75].

LEMMA 2.5. *Let $D^*_{\boldsymbol{q},p}$ be the dual net of a polynomial lattice point set $P_{2^m,s}(\boldsymbol{q}, p)$. Then we have*

$$\frac{1}{2^m}\sum_{i=0}^{2^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_i) = \begin{cases} 1 & \text{if } \boldsymbol{k} \in D^*_{\boldsymbol{q},p}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Mean square weighted $\mathcal{L}_2$ discrepancy. In this section, we study the mean square weighted $\mathcal{L}_2$ discrepancy of scrambled polynomial lattice rules. In [9], Dick and Pillichshammer have derived the Walsh series expansion of the classical $\mathcal{L}_2$ discrepancy. By a slight modification, we can rewrite the expression of the square weighted $\mathcal{L}_2$ discrepancy given in Proposition 1.1 as follows.

PROPOSITION 3.1. *For any point set $P_{N,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and any sequence $\boldsymbol{\gamma} = (\gamma_u)_{u \subseteq I_s}$ of weights, we have*

$$\mathcal{L}^2_{2,N,\boldsymbol{\gamma}}(P_{N,s}) = \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \sum_{\boldsymbol{k}_u, \boldsymbol{l}_u \in \mathbb{N}_0^{|u|} \setminus \{\boldsymbol{0}\}} r_u(\boldsymbol{k}_u, \boldsymbol{l}_u) \frac{1}{N^2} \sum_{n,n'=0}^{N-1} \mathrm{wal}_{\boldsymbol{k}_u}(\boldsymbol{x}_{n,u}) \mathrm{wal}_{\boldsymbol{l}_u}(\boldsymbol{x}_{n',u}),$$

where $\boldsymbol{k}_u = (k_j)_{j \in u}$, $\boldsymbol{l}_u = (l_j)_{j \in u}$, $r_u(\boldsymbol{k}_u, \boldsymbol{l}_u) = \prod_{j \in u} r(k_j, l_j)$. Further, we have $r(k,l) = r(l,k)$, and for non-negative integers $0 \le l \le k$ with dyadic expansions $k = 2^{a_1-1} + \cdots + 2^{a_v-1}$ with $a_1 > \cdots > a_v > 0$ and $l = 2^{b_1-1} + \cdots + 2^{b_w-1}$ with $b_1 > \cdots > b_w > 0$, we have

$$r(k,l) = \begin{cases} \frac{1}{3} & \text{if } k = l = 0, \\ \frac{1}{2^{a_1+2}} & \text{if } v = 1 \text{ and } l = 0, \\ -\frac{1}{2^{a_1+a_2+2}} & \text{if } v = 2 \text{ and } l = 0, \\ -\frac{1}{2^{a_1+a_2+2}} & \text{if } v = w+2 > 2 \text{ and } a_3 = b_1, \dots, a_v = b_w, \\ \frac{1}{3 \cdot 4^{a_1}} & \text{if } k = l > 0, \\ \frac{1}{2^{a_1+b_1+2}} & \text{if } v = w, a_1 \neq b_1 \text{ and } a_2 = b_2, \dots, a_v = b_v, \\ 0 & \text{otherwise.} \end{cases}$$

The next corollary provides an expression for the mean square weighted $\mathcal{L}_2$ discrepancy of scrambled polynomial lattice rules.

COROLLARY 3.2. *For a polynomial lattice point set $P_{2^m,s}(\boldsymbol{q}, p)$, we have*

$$\mathbb{E}[\mathcal{L}^2_{2,2^m,\boldsymbol{\gamma}}(P_{2^m,s}(\boldsymbol{q},p))] = \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q},p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}),$$

where the expectation is taken with respect to all the possible scrambled point sets of $P_{2^m,s}(\boldsymbol{q},p)$. Further, we denote by $(\boldsymbol{k}_v, \boldsymbol{0})$ the vector from $\mathbb{N}_0^s$ with all the components whose indices are not in v replaced by zero, and we have $\psi(k) = 1/4^{a_1}$ for $k \in \mathbb{N}$ with dyadic expansion $k = 2^{a_1-1} + \cdots + 2^{a_v-1}$ with $a_1 > \cdots > a_v > 0$, $\psi(0) = 1$ and $\psi(\boldsymbol{k}) = \prod_{j=1}^s \psi(k_j)$.

Proof. Let $y, y' \in [0,1)$ be two points obtained by applying Owen's scrambling to the points $x, x' \in [0,1)$. From Owen's lemma [8, Lemma 13.3], we have

$$\mathbb{E}[\mathrm{wal}_k(y)\mathrm{wal}_l(y')] = 0, \tag{3.1}$$

whenever $k \neq l$. In the following, we denote by $y_{n,j}$ the point obtained by applying Owen's scrambling to the point $x_{n,j}$. Using Proposition 2.3, Proposition 3.1, (3.1)

and the linearity of expectation, we have

$$\begin{aligned}
&\mathbb{E}[\mathcal{L}^2_{2,2^m,\boldsymbol{\gamma}}(P_{2^m,s}(\boldsymbol{q},p))] \\
&= \sum_{\emptyset\neq u\subseteq I_s} \gamma_u \sum_{\boldsymbol{k}_u,\boldsymbol{l}_u\in\mathbb{N}_0^{|u|}\setminus\{\boldsymbol{0}\}} r_u(\boldsymbol{k}_u,\boldsymbol{l}_u)\frac{1}{2^{2m}}\sum_{n,n'=0}^{2^m-1}\prod_{j\in u}\mathbb{E}[\mathrm{wal}_{k_j}(y_{n,j})\mathrm{wal}_{l_j}(y_{n',j})] \\
&= \sum_{\emptyset\neq u\subseteq I_s} \gamma_u \sum_{\boldsymbol{k}_u\in\mathbb{N}_0^{|u|}\setminus\{\boldsymbol{0}\}} r_u(\boldsymbol{k}_u,\boldsymbol{k}_u)\frac{1}{2^{2m}}\sum_{n,n'=0}^{2^m-1}\prod_{j\in u}\mathbb{E}[\mathrm{wal}_{k_j}(y_{n,j}\oplus y_{n',j})] \\
&= \sum_{\emptyset\neq u\subseteq I_s} \gamma_u \sum_{\emptyset\neq v\subseteq u}\frac{1}{3^{|u\setminus v|}}\sum_{\boldsymbol{k}_v\in\mathbb{N}^{|v|}} r_v(\boldsymbol{k}_v,\boldsymbol{k}_v)\frac{1}{2^{2m}}\sum_{n,n'=0}^{2^m-1}\prod_{j\in v}\mathbb{E}[\mathrm{wal}_{k_j}(y_{n,j}\oplus y_{n',j})].
\end{aligned}$$

Now we need to introduce the following notations. For $\boldsymbol{l}_v=(l_j)_{j\in v}\in\mathbb{N}^{|v|}$, we define a set $\mathcal{B}_{\boldsymbol{l}_v}$ as

$$\mathcal{B}_{\boldsymbol{l}_v}=\{(k_j)_{j\in v}\in\mathbb{N}_0^{|v|} : b^{l_j-1}\leq k_j<b^{l_j} \text{ for } j\in v\}.$$

We denote by $\sigma_{\boldsymbol{l}_v}$ a sum of $r_v(\boldsymbol{k}_v,\boldsymbol{k}_v)$ over all $\boldsymbol{k}_v\in\mathcal{B}_{\boldsymbol{l}_v}$. We have

$$\begin{aligned}
\sigma_{\boldsymbol{l}_v} &= \sum_{\boldsymbol{k}_v\in\mathcal{B}_{\boldsymbol{l}_v}} r_v(\boldsymbol{k}_v,\boldsymbol{k}_v) \\
&= \sum_{\boldsymbol{k}_v\in\mathcal{B}_{\boldsymbol{l}_v}}\prod_{j\in v} r(k_j,k_j) \\
&= \prod_{j\in v}\sum_{k_j=2^{l_j-1}}^{2^{l_j}-1} r(k_j,k_j) \\
&= \prod_{j\in v}\frac{2^{l_j}-2^{l_{j-1}}}{3\cdot 4^{l_j}} \\
&= \frac{1}{3^{|v|}\cdot 2^{|v|+|\boldsymbol{l}_v|_1}},
\end{aligned}$$

where $|\boldsymbol{l}_v|_1:=\sum_{j\in v} l_j$. Further, we introduce a so-called gain coefficient, which is independent of the choice of $\boldsymbol{k}_v\in\mathcal{B}_{\boldsymbol{l}_v}$,

$$\begin{aligned}
G_{\boldsymbol{l}_v} &= \frac{1}{2^{2m}}\sum_{n,n'=0}^{2^m-1}\prod_{j\in v}\mathbb{E}[\mathrm{wal}_{k_j}(y_{n,j}\oplus y_{n',j})] \\
&= 2^{|v|-|\boldsymbol{l}_v|_1}\sum_{\substack{\boldsymbol{k}_v\in\mathcal{B}_{\boldsymbol{l}_v}\\(\boldsymbol{k}_v,\boldsymbol{0})\in D^*_{\boldsymbol{q},p}}} 1,
\end{aligned}$$

where the last equality is appeared in the proof of [8, Corollary 13.7]. Using these

notations and their values, we have

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}^2_{2,2^m,\boldsymbol{\gamma}}(P_{2^m,s}(\boldsymbol{q},p))] \\
&= \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \sum_{\emptyset \neq v \subseteq u} \frac{1}{3^{|u \setminus v|}} \sum_{\boldsymbol{l}_v \in \mathbb{N}^{|v|}} \sum_{\boldsymbol{k}_v \in \mathcal{B}_{\boldsymbol{l}_v}} r_v(\boldsymbol{k}_v, \boldsymbol{k}_v) \frac{1}{2^{2m}} \sum_{n,n'=0}^{2^m-1} \prod_{j \in v} \mathbb{E}[\mathrm{wal}_{k_j}(y_{n,j} \ominus y_{n',j})] \\
&= \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \sum_{\emptyset \neq v \subseteq u} \frac{1}{3^{|u \setminus v|}} \sum_{\boldsymbol{l}_v \in \mathbb{N}^{|v|}} G_{\boldsymbol{l}_v} \sigma_{\boldsymbol{l}_v} \\
&= \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\boldsymbol{l}_v \in \mathbb{N}^{|v|}} \frac{1}{4^{|\boldsymbol{l}_v|_1}} \sum_{\substack{\boldsymbol{k}_v \in \mathcal{B}_{\boldsymbol{l}_v} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q},p}}} 1 \\
&= \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q},p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}).
\end{aligned}
$$

Hence, the result follows. □

We denote the sum in Corollary 3.2 by

$$
B(\boldsymbol{q},\boldsymbol{\gamma}) = \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q},p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}). \tag{3.2}
$$

Using the property of dual net $D^*_{\boldsymbol{q},p}$ shown in Lemma 2.5, we can derive a more computable form of $B(\boldsymbol{q},\boldsymbol{\gamma})$. In the following, we write $\log_2$ for the logarithm in base 2 and we set $2^{\lfloor \log_2 0 \rfloor} = 0$.

Lemma 3.3. *Let $B(\boldsymbol{q},\boldsymbol{\gamma})$ be given by (3.2). Then we have*

$$
B(\boldsymbol{q},\boldsymbol{\gamma}) = -\sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} + \frac{1}{2^m} \sum_{n=0}^{2^m-1} \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \prod_{j \in u} \phi(x_{n,j}), \tag{3.3}
$$

where for $x \in [0,1)$ we set

$$
\phi(x) = \frac{1 - 2^{\lfloor \log_2 x \rfloor}}{2}.
$$

In particular, in case of the so-called product weights, that is, $\gamma_u = \prod_{j \in u} \gamma_j$, we have

$$
B(\boldsymbol{q},\boldsymbol{\gamma}) = -\prod_{j=1}^{s} \left(1 + \frac{\gamma_j}{3}\right) + \frac{1}{2^m} \sum_{n=0}^{2^m-1} \prod_{j=1}^{s} \left[1 + \gamma_j \phi(x_{n,j})\right].
$$

Proof. Applying Lemma 2.5 to $B(\boldsymbol{q},\boldsymbol{\gamma})$, we have

$$
\begin{aligned}
B(\boldsymbol{q},\boldsymbol{\gamma}) &= \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\boldsymbol{k}_v \in \mathbb{N}^{|v|}} \psi(\boldsymbol{k}_v, \boldsymbol{0}) \frac{1}{2^m} \sum_{n=0}^{2^m-1} \mathrm{wal}_{(\boldsymbol{k}_v,\boldsymbol{0})}(\boldsymbol{x}_n) \\
&= \frac{1}{2^m} \sum_{n=0}^{2^m-1} \sum_{\emptyset \neq u \subseteq I_s} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \prod_{j \in v} \left[\sum_{k_j=1}^{\infty} \psi(k_j) \mathrm{wal}_{k_j}(x_{n,j}) \right].
\end{aligned}
$$

For the innermost sum, we have by following the similar line as the proof of [1, Theorem 7.3]

$$\sum_{k=1}^{\infty}\psi(k)\mathrm{wal}_k(x)=\sum_{l=1}^{\infty}\frac{1}{4^l}\sum_{k=2^{l-1}}^{2^l-1}\mathrm{wal}_k(x)$$
$$=\frac{1-3\cdot 2^{\lfloor\log_2(x)\rfloor}}{2},$$

Then, we have

$$\begin{aligned}B(\boldsymbol{q},\boldsymbol{\gamma})&=\frac{1}{2^m}\sum_{n=0}^{2^m-1}\sum_{\emptyset\neq u\subseteq I_s}\frac{\gamma_u}{3^{|u|}}\sum_{\emptyset\neq v\subseteq u}\prod_{j\in v}\left(\frac{1-3\cdot 2^{\lfloor\log_2(x_{n,j})\rfloor}}{2}\right)\\&=\frac{1}{2^m}\sum_{n=0}^{2^m-1}\sum_{\emptyset\neq u\subseteq I_s}\frac{\gamma_u}{3^{|u|}}\left[-1+\prod_{j\in u}\left(1+\frac{1-3\cdot 2^{\lfloor\log_2(x_{n,j})\rfloor}}{2}\right)\right],\end{aligned}$$

which proves the first part of the lemma.

Next in case of $\gamma_v=\prod_{j\in v}\gamma_j$, by letting $\gamma_\emptyset=1$, we have

$$\begin{aligned}B(\boldsymbol{q},\boldsymbol{\gamma})&=-\sum_{u\subseteq I_s}\prod_{j\in u}\frac{\gamma_j}{3}+\frac{1}{2^m}\sum_{n=0}^{2^m-1}\sum_{u\subseteq I_s}\prod_{j\in u}\gamma_j\phi(x_{n,j})\\&=-\prod_{j=1}^{s}\left(1+\frac{\gamma_j}{3}\right)+\frac{1}{2^m}\sum_{n=0}^{2^m-1}\prod_{j=1}^{s}\left[1+\gamma_j\phi(x_{n,j})\right].\end{aligned}$$

Hence, the result for the second part of the lemma follows. □

Since we have the following recursion in the inner sum of (3.3)

$$\sum_{\emptyset\neq u\subseteq I_r}\gamma_u\prod_{j\in u}\phi(x_{n,j})=\gamma_{I_r}\prod_{j=1}^{r}\phi(x_{n,j})+\sum_{\emptyset\neq u\subseteq I_{r-1}}\left(1+\frac{\gamma_{u\cup\{r\}}}{\gamma_u}\phi(x_{r,j})\right)\gamma_u\prod_{j\in u}\phi(x_{n,j}),$$

for $1\le r\le s$, the computational complexity of $B(\boldsymbol{q},\boldsymbol{\gamma})$ with the general weights is $O(2^{ms})$. In case of the product weights, on the other hand, the computational complexity of $B(\boldsymbol{q},\boldsymbol{\gamma})$ reduces to $O(s2^m)$.

4. Construction of polynomial lattice rules. In this section, we first show to how to find a vector $\boldsymbol{q}$ by using the CBC construction algorithm. Then, we prove that an upper bound on $B(\boldsymbol{q},\boldsymbol{\gamma})$ obtained by our algorithm converges at almost the best possible rate of $N^{-2+\delta}$ for all $\delta>0$. Finally we end this section with the discussion about the strong tractability of our algorithm.

We restrict each polynomial q_j such that $\deg(q_j)<m$ and $q_j\neq 0$. In the following, we denote by R_m the set of all the non-zero polynomials over $\mathbb{F}_2$ with degree less than m, i.e.,

$$R_m=\{q\in\mathbb{F}_2[x]:\deg(q)<m\text{ and }q\neq 0\}.$$

It is clear that $|R_m|=2^m-1$. Further, we write $\boldsymbol{q}_j=(q_1,\dots,q_j)$ for $1\le j\le s$. The CBC construction proceeds as follows.

ALGORITHM 4.1. *For $m,s\in\mathbb{N}$ and any sequence $\boldsymbol{\gamma}=(\gamma_u)_{u\subseteq I_s}$, we proceed as follows.*

1. *Choose an irreducible polynomial* $p \in \mathbb{F}_2[x]$ *with* $\deg(p) = m$.
2. *Set* $q_1^* = 1$.
3. *For* $j = 2, \cdots, s$, *find* q_j^* *by minimizing* $B((\boldsymbol{q}_{j-1}^*, q_j), \boldsymbol{\gamma})$ *as a function of* $q_j \in R_m$ *where*

$$B((\boldsymbol{q}_{j-1}^*, q_j), \boldsymbol{\gamma}) = \sum_{\emptyset \neq u \subseteq I_j} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \neq v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{(\boldsymbol{q}_{j-1}^*, q_j), p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}).$$

The next theorem provides an upper bound on $B(\boldsymbol{q}_j, \boldsymbol{\gamma})$ for the thus constructed polynomials $\boldsymbol{q}_j^*$ for $1 \leq j \leq s$, which converges at almost the best possible rate of $N^{-2+\delta}$ for all $\delta > 0$. In the proof of the theorem, we use Jensen's inequality, which states that for any sequence (a_i) of non-negative real numbers we have

$$\left(\sum a_i\right)^\lambda \leq \sum a_i^\lambda,$$

for any $0 < \lambda \leq 1$. Our proof is similar to the proof of [6, Theorem 4.4] but requires the slightly more complicated argument.

THEOREM 4.2. *Let* $p \in \mathbb{F}_2[x]$ *be irreducible polynomial with* $\deg(p) = m$. *Suppose that* $\boldsymbol{q}_s^* \in R_m^s$ *is constructed by using Algorithm 4.1. Then for any* $j = 1, \ldots, s$ *we have*

$$B(\boldsymbol{q}_j^*, \boldsymbol{\gamma}) \leq \frac{1}{(2^m - 1)^{1/\lambda}} \left[\sum_{\emptyset \neq u \subseteq I_j} \left(\frac{\gamma_u}{3^{|u|}}\right)^\lambda \left(-1 + \left(\frac{2^{2\lambda} - 1}{2^{2\lambda} - 2}\right)^{|u|}\right)\right]^{1/\lambda}, \tag{4.1}$$

and for $\emptyset \neq v \subseteq I_j$

$$\sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q}_j^*, p}}} \psi^\lambda(\boldsymbol{k}_v, \boldsymbol{0}) \leq \frac{1}{(2^m - 1)(2^{2\lambda} - 2)^{|v|}}, \tag{4.2}$$

for $1/2 < \lambda \leq 1$.

Proof. We prove the theorem by induction on j. For $j = 1$, we have

$$\begin{aligned}
B(q_1^*, \boldsymbol{\gamma}) &= \frac{\gamma_{\{1\}}}{3} \sum_{\substack{k=1 \\ 2^m | k}}^{\infty} \psi(k) \\
&\leq \frac{\gamma_{\{1\}}}{3} \left(\sum_{a=1}^{\infty} \sum_{\substack{k=2^{a-1} \\ 2^m | k}}^{2^a - 1} \psi^\lambda(k)\right)^{1/\lambda} \\
&= \frac{\gamma_{\{1\}}}{3} \left(\sum_{a=m+1}^{\infty} 2^{a-m-1} \cdot 2^{-2a\lambda}\right)^{1/\lambda} \\
&= \frac{\gamma_{\{1\}}}{3} \left[\frac{1}{2^{2\lambda m}(2^{2\lambda} - 2)}\right]^{1/\lambda} \\
&\leq \frac{\gamma_{\{1\}}}{3} \left[\frac{1}{(2^m - 1)(2^{2\lambda} - 2)}\right]^{1/\lambda}.
\end{aligned}$$

for $1/2 < \lambda \le 1$, where we used Jensen's inequality for the first inequality. It is clear that the proof of (4.2) is included in this calculation. Hence, the result follows for $j = 1$.

Next, let the statement of the theorem be true for some $j \ge 2$. Then it is enough to show that the statement is also true for the $(j+1)$-th component. In the following, we classify each subset $u \subseteq I_{j+1}$ according to whether u includes the component $\{j+1\}$. If u includes $\{j+1\}$, we further classify each subset $v \subseteq u$ according to whether v includes the component $\{j+1\}$. Then we have

$$\begin{aligned}
& B((\boldsymbol{q}_j^*, q_{j+1}), \boldsymbol{\gamma}) \\
&= \sum_{\emptyset \ne u \subseteq I_{j+1}} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \ne v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{(\boldsymbol{q}_j^*, q_{j+1}), p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}) \\
&= \sum_{\emptyset \ne u \subseteq I_j} \frac{\gamma_u}{3^{|u|}} \sum_{\emptyset \ne v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q}_j^*, p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}) \\
&\quad + \sum_{u \subseteq I_j} \frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \sum_{\emptyset \ne v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q}_j^*, p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}) \\
&\quad + \sum_{u \subseteq I_j} \frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \sum_{v \subseteq u} \sum_{\substack{(\boldsymbol{k}_v, k_{j+1}) \in \mathbb{N}^{|v|+1} \\ (\boldsymbol{k}_v, k_{j+1}, \boldsymbol{0}) \in D^*_{(\boldsymbol{q}_j^*, q_{j+1}), p}}} \psi(\boldsymbol{k}_v, k_{j+1}, \boldsymbol{0}) \\
&= B(\boldsymbol{q}_j^*, \boldsymbol{\gamma}) + \theta_1 + \theta_2(q_{j+1}),
\end{aligned}$$

where we have defined

$$\theta_1 := \sum_{u \subseteq I_j} \frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \sum_{\emptyset \ne v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q}_j^*, p}}} \psi(\boldsymbol{k}_v, \boldsymbol{0}),$$

and

$$\theta_2(q_{j+1}) := \sum_{u \subseteq I_j} \frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \sum_{v \subseteq u} \sum_{\substack{(\boldsymbol{k}_v, k_{j+1}) \in \mathbb{N}^{|v|+1} \\ (\boldsymbol{k}_v, k_{j+1}, \boldsymbol{0}) \in D^*_{(\boldsymbol{q}_j^*, q_{j+1}), p}}} \psi(\boldsymbol{k}_v, k_{j+1}, \boldsymbol{0}).$$

We note that θ_2 is a function of q_{j+1}, while θ_1 is not.

Using Jensen's inequality and (4.2), we obtain an upper bound on θ_1 as

$$\begin{aligned}
\theta_1^\lambda &\le \sum_{u \subseteq I_j} \left(\frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \right)^\lambda \sum_{\emptyset \ne v \subseteq u} \sum_{\substack{\boldsymbol{k}_v \in \mathbb{N}^{|v|} \\ (\boldsymbol{k}_v, \boldsymbol{0}) \in D^*_{\boldsymbol{q}_j^*, p}}} \psi^\lambda(\boldsymbol{k}_v, \boldsymbol{0}) \\
&\le \sum_{u \subseteq I_j} \left(\frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \right)^\lambda \sum_{\emptyset \ne v \subseteq u} \frac{1}{(2^m - 1)(2^{2\lambda} - 2)^{|v|}} \\
&= \frac{1}{2^m - 1} \sum_{u \subseteq I_j} \left(\frac{\gamma_{u \cup \{j+1\}}}{3^{|u|+1}} \right)^\lambda \left(-1 + \left(\frac{2^{2\lambda} - 1}{2^{2\lambda} - 2} \right)^{|u|} \right).
\end{aligned}$$

In order to obtain an upper bound on $\theta_2(q_{j+1}^*)$, we employ the averaging argument as in the proof of [6, Theorem 4.4]. Since we choose q_{j+1} which minimizes $\theta_2(q_{j+1})$ according to Algorithm 4.1, we have

$$\begin{aligned}\theta_2^\lambda(q_{j+1}^*) &\le \frac{1}{2^m-1}\sum_{q_{j+1}\in R_m}\theta_2^\lambda(q_{j+1}) \\ &\le \sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda \sum_{v\subseteq u}\mu_{v,j+1},\end{aligned}$$

where we have defined

$$\mu_{v,j+1} := \frac{1}{2^m-1}\sum_{q_{j+1}\in R_m}\sum_{\substack{(\boldsymbol{k}_v,k_{j+1})\in\mathbb{N}^{|v|+1}\\ (\boldsymbol{k}_v,k_{j+1},\boldsymbol{0})\in D^*_{(\boldsymbol{q}_j^*,q_{j+1}),p}}}\psi^\lambda(\boldsymbol{k}_v,k_{j+1},\boldsymbol{0}).$$

Due to the condition of $(\boldsymbol{k}_v,k_{j+1},\boldsymbol{0})\in D^*_{(\boldsymbol{q}_j^*,q_{j+1}),p}$, $\boldsymbol{k}_v$ and k_{j+1} must satisfy

$$\mathrm{tr}_m(\boldsymbol{k}_v)\cdot\boldsymbol{q}_v^* + \mathrm{tr}_m(k_{j+1})\cdot q_{j+1} = 0 \pmod p.$$

If k_{j+1} is a multiple of 2^m, we always have $\mathrm{tr}_m(k_{j+1})=0$ and the above equation becomes independent of q_{j+1}. Otherwise if k_{j+1} is not a multiple of 2^m, we have $\mathrm{tr}_m(k_{j+1})\neq 0$ and the term $\mathrm{tr}_m(k_{j+1})\cdot q_{j+1}$ cannot be a multiple of p. Thus, we have

$$\mu_{v,j+1} = \sum_{\substack{k_{j+1}=1\\ 2^m|k_{j+1}}}^{\infty}\psi^\lambda(k_{j+1})\sum_{\substack{\boldsymbol{k}_v\in\mathbb{N}^{|v|}\\ \mathrm{tr}_m(\boldsymbol{k}_v)\cdot\boldsymbol{q}_v=0 \pmod p}}\psi^\lambda(\boldsymbol{k}_v) \tag{4.3}$$

$$+\frac{1}{2^m-1}\sum_{\substack{k_{j+1}=1\\ 2^m\nmid k_{j+1}}}^{\infty}\psi^\lambda(k_{j+1})\sum_{\substack{\boldsymbol{k}_v\in\mathbb{N}^{|v|}\\ \mathrm{tr}_m(\boldsymbol{k}_v)\cdot\boldsymbol{q}_v\neq 0 \pmod p}}\psi^\lambda(\boldsymbol{k}_v). \tag{4.4}$$

We obtained the second equality by applying the definition of the dual net and by classifying whether k_{j+1} is a multiple of p or not. In (4.3), we have

$$\begin{aligned}\sum_{\substack{k_{j+1}=1\\ 2^m|k_{j+1}}}^{\infty}\psi^\lambda(k_{j+1}) &= \sum_{a=m+1}^{\infty}\sum_{\substack{k_{j+1}=2^{a-1}\\ 2^m|k_{j+1}}}^{2^a-1}\psi^\lambda(k_{j+1}) \\ &= \sum_{a=m+1}^{\infty}2^{a-m-1}\cdot 2^{-2\lambda a} \\ &= \frac{1}{2^{m+1}}\sum_{a=m+1}^{\infty}2^{(1-2\lambda)a}.\end{aligned}$$

In (4.4), we have

$$\begin{aligned}
&\frac{1}{2^m-1}\sum_{\substack{k_{j+1}=1\\ 2^m\nmid k_{j+1}}}^{\infty}\psi^{\lambda}(k_{j+1})\\
&=\frac{1}{2^m-1}\sum_{a=1}^{m}\sum_{k_{j+1}=2^{a-1}}^{2^a-1}\psi^{\lambda}(k_{j+1})+\frac{1}{2^m-1}\sum_{a=m+1}^{\infty}\sum_{\substack{k_{j+1}=2^{a-1}\\ 2^m\nmid k_{j+1}}}^{2^a-1}\psi^{\lambda}(k_{j+1})\\
&=\frac{1}{2^m-1}\sum_{a=1}^{m}2^{a-1}\cdot 2^{-2\lambda a}+\frac{1}{2^m-1}\sum_{a=m+1}^{\infty}(2^{a-1}-2^{a-m-1})\cdot 2^{-2\lambda a}\\
&=\frac{1}{2(2^m-1)}\sum_{a=1}^{m}2^{(1-2\lambda)a}+\frac{1}{2^{m+1}}\sum_{a=m+1}^{\infty}2^{(1-2\lambda)a}.
\end{aligned}$$

Hence, $\mu_{v,j+1}$ is upper-bounded by

$$\begin{aligned}
&\mu_{v,j+1}\\
&=\frac{1}{2^{m+1}}\sum_{a=m+1}^{\infty}2^{(1-2\lambda)a}\sum_{\substack{\boldsymbol{k}_v\in\mathbb{N}^{|v|}\\ \mathrm{tr}_m(\boldsymbol{k}_v)\cdot\boldsymbol{q}_v=0 \pmod p}}\psi^{\lambda}(\boldsymbol{k}_v)\\
&\quad+\left(\frac{1}{2(2^m-1)}\sum_{a=1}^{m}2^{(1-2\lambda)a}+\frac{1}{2^{m+1}}\sum_{a=m+1}^{\infty}2^{(1-2\lambda)a}\right)\sum_{\substack{\boldsymbol{k}_v\in\mathbb{N}^{|v|}\\ \mathrm{tr}_m(\boldsymbol{k}_v)\cdot\boldsymbol{q}_v\neq 0 \pmod p}}\psi^{\lambda}(\boldsymbol{k}_v)\\
&\le\left(\frac{1}{2(2^m-1)}\sum_{a=1}^{m}2^{(1-2\lambda)a}+\frac{1}{2^{m+1}}\sum_{a=m+1}^{\infty}2^{(1-2\lambda)a}\right)\sum_{\boldsymbol{k}_v\in\mathbb{N}^{|v|}}\psi^{\lambda}(\boldsymbol{k}_v)\\
&\le\frac{1}{2(2^m-1)}\sum_{a=1}^{\infty}2^{(1-2\lambda)a}\sum_{\boldsymbol{k}_v\in\mathbb{N}^{|v|}}\psi^{\lambda}(\boldsymbol{k}_v).
\end{aligned}$$

In the last expression, we have

$$\sum_{a=1}^{\infty}2^{(1-2\lambda)a}=\frac{2}{2^{2\lambda}-2},$$

and

$$\begin{aligned}
\sum_{\boldsymbol{k}_v\in\mathbb{N}^{|v|}}\psi^{\lambda}(\boldsymbol{k}_v)&=\left[\sum_{k=1}^{\infty}\psi^{\lambda}(k)\right]^{|v|}\\
&=\left[\sum_{a=1}^{\infty}\sum_{k=2^{a-1}}^{2^a-1}\psi^{\lambda}(k)\right]^{|v|}\\
&=\left[\sum_{a=1}^{\infty}2^{a-1}\cdot 2^{-2\lambda a}\right]^{|v|}\\
&=\frac{1}{(2^{\lambda}-2)^{|v|}}.
\end{aligned}$$

Thus, we obtain

$$\mu_{v,j+1} \le \frac{1}{(2^m-1)(2^{2\lambda}-2)^{|v|+1}},$$

which implies

$$\sum_{\substack{(\boldsymbol{k}_v,k_{j+1})\in\mathbb{N}^{|v|+1}\\ (\boldsymbol{k}_v,k_{j+1},\boldsymbol{0})\in D^*_{\boldsymbol{q}^*_{j+1},p}}} \psi^\lambda(\boldsymbol{k}_v,k_{j+1},\boldsymbol{0}) \le \frac{1}{(2^m-1)(2^{2\lambda}-2)^{|v|+1}},$$

for any $v\subseteq I_j$. By combining this result with (4.2) for $\emptyset\neq v\subseteq I_s$, which is supposed to be true by induction, we have (4.2) for any $\emptyset\subset v\subseteq I_{j+1}$.

We return to obtain an upper bound on $\theta_2(q^*_{j+1})$

$$\begin{aligned}
\theta_2^\lambda(q^*_{j+1}) &\le \sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda \sum_{v\subseteq u}\mu_{v,j+1} \\
&\le \frac{1}{2^m-1}\sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda \sum_{v\subseteq u}\frac{1}{(2^{2\lambda}-2)^{|v|+1}} \\
&= \frac{1}{2^m-1}\sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda \frac{1}{2^{2\lambda}-2}\left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}.
\end{aligned}$$

Finally, by using Jensen's inequality again, we have

$$\begin{aligned}
& B^\lambda((\boldsymbol{q}^*_j,q^*_{j+1}),\boldsymbol{\gamma}) \\
&= \left(B(\boldsymbol{q}^*_j,\boldsymbol{\gamma})+\theta_1+\theta_2(q^*_{j+1})\right)^\lambda \\
&\le B^\lambda(\boldsymbol{q}^*_j,\boldsymbol{\gamma})+\theta_1^\lambda+\theta_2^\lambda(q^*_{j+1}) \\
&\le \frac{1}{2^m-1}\sum_{\emptyset\neq u\subseteq I_j}\left(\frac{\gamma_u}{3^{|u|}}\right)^\lambda\left(-1+\left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}\right) \\
&\quad +\frac{1}{2^m-1}\sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda\left(-1+\left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}\right) \\
&\quad +\frac{1}{2^m-1}\sum_{u\subseteq I_j}\left(\frac{\gamma_{u\cup\{j+1\}}}{3^{|u|+1}}\right)^\lambda\frac{1}{2^{2\lambda}-2}\left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|} \\
&= \frac{1}{2^m-1}\sum_{\emptyset\neq u\subseteq I_{j+1}}\left(\frac{\gamma_u}{3^{|u|}}\right)^\lambda\left(-1+\left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}\right),
\end{aligned}$$

for $1/2<\lambda\le 1$, which proves (4.1). Hence, the result follows. □

Remark 4.3. *For $j=1$, we can obtain the exact value for $B(q^*_1,\boldsymbol{\gamma})$ as*

$$\begin{aligned}
B(q^*_1,\boldsymbol{\gamma}) &= \frac{\gamma_{\{1\}}}{3}\sum_{\substack{k=1\\ 2^m|k}}^\infty \psi(k) \\
&= \frac{\gamma_{\{1\}}}{3}\sum_{a=m+1}^\infty 2^{a-m-1}\cdot 2^{-2a} \\
&= \frac{\gamma_{\{1\}}}{6}\cdot\frac{1}{2^{2m}}.
\end{aligned}$$

We avoid this calculation in the above proof so as to prove both (4.1) and (4.2). Since the lower bound on the $\mathcal{L}_2$ discrepancy is given as (1.2), this achieves the best possible rate of convergence. As one-dimensional polynomial lattice point set consists of the equi-distributed points $x_n = n/b^m$, $n = 0, \dots, 2^m - 1$, another QMC point sets such as Sobol' and Niederreiter sequences constructed over $\mathbb{F}_2$ also give the same value.

Here we mention about the strong tractability of our construction algorithm. Let us consider the inverse of the mean square weighted $\mathcal{L}_2$ discrepancy which is defined as follows

$$N(s,\epsilon) = \min\{N \in \mathbb{N} : \mathbb{E}[\mathcal{L}^2_{2,N,\boldsymbol{\gamma}}(P_{N,s})] \le \epsilon \mathbb{E}[\mathcal{L}^2_{2,0,\boldsymbol{\gamma}}(P_{0,s})]\}.$$

We say that the mean square weighted $\mathcal{L}_2$ discrepancy is strongly tractable if there exist non-negative constants C and β such that

$$N(s,\epsilon) \le C\epsilon^{-\beta},$$

where C depends neither on ϵ and s.

Corollary 4.4. *Assume that the weights $\boldsymbol{\gamma}$ satisfy the condition*

$$B_{\boldsymbol{\gamma},\lambda} := \sup_{s\in\mathbb{N}} \frac{\left[\sum_{\emptyset\ne u\subseteq I_s} \left(\frac{\gamma_u}{3^{|u|}}\right)^\lambda \left(-1 + \left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}\right)\right]^{1/\lambda}}{\sum_{\emptyset\ne u\subseteq I_s} \frac{\gamma_u}{3^{|u|}}} < \infty,$$

for some λ. Then the mean square weighted $\mathcal{L}_2$ discrepancy is strongly tractable.

Proof. For empty point set $P_{0,s}$, we have

$$\begin{aligned}\mathbb{E}[\mathcal{L}^2_{2,0,\boldsymbol{\gamma}}(P_{0,s})] &= \sum_{\emptyset\ne u\subseteq I_s} \gamma_u \prod_{j\in u}\int_0^1 t_j^2\,dt_j \\ &= \sum_{\emptyset\ne u\subseteq I_s} \frac{\gamma_u}{3^{|u|}}.\end{aligned}$$

For a polynomial lattice point set $P_{2^m,s}$ constructed by Algorithm 4.1, we have

$$\begin{aligned}\mathbb{E}[\mathcal{L}^2_{2,2^m,\boldsymbol{\gamma}}(P_{2^m,s})] &\le \frac{1}{(2^m-1)^{1/\lambda}}\left[\sum_{\emptyset\ne u\subseteq I_s} \left(\frac{\gamma_u}{3^{|u|}}\right)^\lambda \left(-1 + \left(\frac{2^{2\lambda}-1}{2^{2\lambda}-2}\right)^{|u|}\right)\right]^{1/\lambda} \\ &\le \frac{1}{(2^m-1)^{1/\lambda}} B_{\boldsymbol{\gamma},\lambda} \sum_{\emptyset\ne u\subseteq I_s} \frac{\gamma_u}{3^{|u|}} \\ &= \frac{1}{(2^m-1)^{1/\lambda}} B_{\boldsymbol{\gamma},\lambda} \mathbb{E}[\mathcal{L}^2_{2,0,\boldsymbol{\gamma}}(P_{0,s})].\end{aligned}$$

The last term is smaller than $\epsilon\mathbb{E}[\mathcal{L}^2_{2,0,\boldsymbol{\gamma}}(P_{0,s})]$ if $N = 2^m \ge 1 + B^\lambda_{\boldsymbol{\gamma},\lambda}\epsilon^{-\lambda}$. Thus, the result follows. ∎

5. Numerical experiments. Finally, we demonstrate the performance of our constructed polynomial lattice rules. We focus on the case of the product weights, that is, $\gamma_u = \prod_{j\in u}\gamma_j$, because of their importance in practice and the availability of the fast CBC construction algorithm using fast Fourier transform [24, 25]. Three choices for γ_j are considered here: $\gamma_j = 1$ (unweighted), $\gamma_j = 0.9^j$ and $\gamma_j = 1/j^2$ for $j = 1, \dots, s$.

TABLE 5.1
The mean square weighted $\mathcal{L}_2$ discrepancy for $\gamma_j = 1$.

m	$s=1$		$s=5$		$s=50$		$s=100$	
	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR
4	6.51e-04	6.51e-04	4.83e-02	3.79e-02	3.93e+07	3.91e+07	2.54e+16	2.54e+16
5	1.63e-04	1.63e-04	1.45e-02	1.37e-02	1.96e+07	1.94e+07	1.27e+16	1.27e+16
6	4.07e-05	4.07e-05	5.04e-03	4.29e-03	9.70e+06	9.64e+06	6.35e+15	6.35e+15
7	1.02e-05	1.02e-05	1.27e-03	1.32e-03	4.78e+06	4.77e+06	3.18e+15	3.18e+15
8	2.54e-06	2.54e-06	4.11e-04	4.69e-04	2.36e+06	2.35e+06	1.59e+15	1.59e+15
9	6.36e-07	6.36e-07	1.21e-04	1.38e-04	1.17e+06	1.16e+06	7.94e+14	7.94e+14
10	1.59e-07	1.59e-07	4.01e-05	4.47e-05	5.80e+05	5.70e+05	3.97e+14	3.97e+14
11	3.97e-08	3.97e-08	1.15e-05	1.28e-05	2.89e+05	2.80e+05	1.98e+14	1.98e+14
12	9.93e-09	9.93e-09	3.45e-06	4.41e-06	1.44e+05	1.37e+05	9.92e+13	9.91e+13
13	2.48e-09	2.48e-09	1.17e-06	1.39e-06	7.17e+04	6.69e+04	4.96e+13	4.95e+13
14	6.21e-10	6.21e-10	2.78e-07	4.05e-07	3.56e+04	3.27e+04	2.48e+13	2.48e+13
15	1.55e-10	1.55e-10	7.98e-08	1.31e-07	1.76e+04	1.59e+04	1.24e+13	1.24e+13

TABLE 5.2
The mean square weighted $\mathcal{L}_2$ discrepancy for $\gamma_j = 0.9^j$.

m	$s=1$		$s=5$		$s=50$		$s=100$	
	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR
4	5.86e-04	5.86e-04	2.13e-02	1.72e-02	1.43e+00	1.22e+00	1.48e+00	1.26e+00
5	1.46e-04	1.46e-04	6.25e-03	5.93e-03	6.27e-01	5.16e-01	6.47e-01	5.34e-01
6	3.66e-05	3.66e-05	2.07e-03	1.80e-03	2.47e-01	2.17e-01	2.56e-01	2.25e-01
7	9.16e-06	9.16e-06	5.25e-04	5.41e-04	9.81e-02	8.85e-02	1.02e-01	9.19e-02
8	2.29e-06	2.29e-06	1.64e-04	1.84e-04	3.94e-02	3.52e-02	4.11e-02	3.67e-02
9	5.72e-07	5.72e-07	4.73e-05	5.23e-05	1.60e-02	1.41e-02	1.66e-02	1.47e-02
10	1.43e-07	1.43e-07	1.52e-05	1.70e-05	6.73e-03	5.62e-03	7.02e-03	5.87e-03
11	3.58e-08	3.58e-08	4.29e-06	5.19e-06	2.97e-03	2.26e-03	3.10e-03	2.36e-03
12	8.94e-09	8.94e-09	1.25e-06	1.58e-06	1.25e-03	8.90e-04	1.31e-03	9.33e-04
13	2.24e-09	2.24e-09	4.01e-07	4.85e-07	5.61e-04	3.57e-04	5.86e-04	3.75e-04
14	5.59e-10	5.59e-10	9.89e-08	1.43e-07	2.13e-04	1.41e-04	2.24e-04	1.49e-04
15	1.40e-10	1.40e-10	2.79e-08	4.38e-08	7.84e-05	5.61e-05	8.30e-05	5.91e-05

We compare the performance of our constructed polynomial lattice rules with that of the Sobol' sequences, which is one of the most well-known digital sequences over $\mathbb{F}_2$ [35]. Since the weights emphasize the relative importance of the discrepancy of lower dimensional projections, we use the Sobol' sequences as constructed in [12], which should work as a good competitor.

In Table 5.1-5.3, we show the values of the mean square weighted $\mathcal{L}_2$ discrepancy for the Sobol' sequences and our constructed polynomial lattice point sets, denoted by Sobol' and PLR respectively, with $m = 4, \dots, 15$ and $s = 1, 5, 50, 100$ and different choices for the weights.

As expected from Remark 4.3, we obtain exactly the same values for both the rules for $s = 1$ and achieve the optimal rate of convergence, 2^{-2m}, despite of the choice of the weights. In case of $s = 5$, although the Sobol' sequences provide the slightly better results for large m, the values are comparable. For $s = 50$ and $s = 100$, we obtain almost the same values for both the rules in the unweighted case, while our constructed polynomial lattice rules outperform the Sobol' sequences in other cases.

REFERENCES

[1] J. BALDEAUX, *Higher order nets and sequences*, PhD thesis, The University of New South Wales, 2010.
[2] J. BALDEAUX AND J. DICK, *A construction of polynomial lattice rules with small gain coefficients*, Numer. Math., 119 (2011) pp. 271–297.
[3] R. E. CAFLISCH, W. MOROKOFF AND A. B. OWEN, *Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension*, J. Comput. Finance, 1 (1997) pp. 27–

Table 5.3
The mean square weighted $\mathcal{L}_2$ discrepancy for $\gamma_j = 1/j^2$.

m	$s=1$		$s=5$		$s=50$		$s=100$	
	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR	Sobol'	PLR
4	6.51e-04	6.51e-04	1.84e-03	1.73e-03	2.99e-03	2.47e-03	3.07e-03	2.53e-03
5	1.63e-04	1.63e-04	4.81e-04	4.76e-04	8.63e-04	7.31e-04	8.95e-04	7.50e-04
6	4.07e-05	4.07e-05	1.35e-04	1.28e-04	2.64e-04	2.10e-04	2.78e-04	2.17e-04
7	1.02e-05	1.02e-05	3.53e-05	3.43e-05	7.42e-05	5.98e-05	8.09e-05	6.24e-05
8	2.54e-06	2.54e-06	9.21e-06	9.43e-06	2.23e-05	1.75e-05	2.48e-05	1.84e-05
9	6.36e-07	6.36e-07	2.53e-06	2.51e-06	6.56e-06	4.94e-06	7.37e-06	5.24e-06
10	1.59e-07	1.59e-07	6.94e-07	6.86e-07	1.75e-06	1.41e-06	2.02e-06	1.51e-06
11	3.97e-08	3.97e-08	1.82e-07	1.90e-07	4.87e-07	4.12e-07	5.53e-07	4.43e-07
12	9.93e-09	9.93e-09	4.76e-08	5.00e-08	1.39e-07	1.16e-07	1.62e-07	1.26e-07
13	2.48e-09	2.48e-09	1.29e-08	1.35e-08	4.06e-08	3.40e-08	4.89e-08	3.70e-08
14	6.21e-10	6.21e-10	3.35e-09	3.80e-09	1.29e-08	1.01e-08	1.53e-08	1.10e-08
15	1.55e-10	1.55e-10	8.87e-10	1.01e-09	3.61e-09	2.97e-09	4.37e-09	3.27e-09

46.

[4] H. E. Chrestenson, *A class of generalized Walsh functions*, Pacific J. Math., 5 (1955) pp. 17–31.

[5] R. Cranley and T. N. L. Patterson, *Randomization of number theoretic methods for multiple integration*, SIAM. J. Numer. Anal., 13 (1976) pp. 904–914.

[6] J. Dick, F. Y. Kuo, F. Pillichshammer and I. H. Sloan, *Construction algorithms for polynomial lattice rules for multivariate integration*, Math. Comp., 74 (2005) pp. 1895–1921.

[7] J. Dick, G. Leobacher, and F. Pillichshammer, *Construction algorithms for digital nets with low weighted star discrepancy*, SIAM. J. Numer. Anal., 43 (2005) pp. 76–95.

[8] J. Dick and F. Pillichshammer, *Digital nets and sequences. Discrepancy theory and quasi-Monte Carlo integration*, Cambridge University Press, Cambridge, 2010.

[9] J. Dick and F. Pillichshammer, *Optimal $\mathcal{L}_2$ discrepancy bounds for higher order digital sequences over finite field $\mathbb{F}_2$*, submitted for publication. arXiv:1207.5189

[10] N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc., 65 (1949) pp. 372–414.

[11] F. J. Hickernell, *A generalized discrepancy and quadrature error bound*, Math. Comp., 67 (1998) pp. 299–322.

[12] S. Joe, and F. Y. Kuo, *Constructing Sobol' sequences with better two-dimensional projections*, SIAM J. Sci. Comput., 30 (2008) pp. 2635–2654.

[13] N. M. Korobov, *The approximate computation of multiple integrals/approximate evaluation of repeated integrals*, Dokl. Akad. Nauk SSSR, 124 (1959) pp. 1207–1210.

[14] P. Kritzer, and F. Pillichshammer, *Constructions of general polynomial lattices for multivariate integration*, Bull. Anstral. Math. Soc., 76 (2007) pp. 93–110.

[15] P. Kritzer, and F. Pillichshammer, *On the component by component construction of polynomial lattice point sets for numerical integration in weighted Sobolev spaces*, Unif. Distrib. Theory, 6 (2011) pp. 79–100.

[16] P. L'Ecuyer, *Polynomial integration lattices*, In: Monte Carlo and Quasi-Monte Carlo Methods 2002, H. Niederreiter (Ed.), Springer, Berlin, 2004, pp. 73–98.

[17] C. Lemieux, *Monte Carlo and quasi-Monte Carlo sampling*, Springer Series in Statistics, Springer, New York, 2009.

[18] J. Matoušek, *On the $\mathcal{L}_2$-discrepancy for anchored boxes*, J. Complexity, 14 (1998) pp. 527–556.

[19] J. Matoušek, *Geometric discrepancy. An illustrated guide*, Algorithms and Combinatorics, Springer, Berlin, 1999.

[20] H. Niederreiter, *Random number generation and quasi-Monte Carlo methods*, CBMS-NSF Series in Applied Mathematics, vol. 63, SIAM, Philadelphia, 1992.

[21] H. Niederreiter, *Low-discrepancy point sets obtained by digital constructions over finite fields*, Czechoslovak Math. J., 42 (1992) pp. 143–166.

[22] S. Ninomiya and S. Tezuka, *Toward real-time pricing of complex financial derivatives*, Appl. Math. Finance, 3 (1996) pp. 1–20.

[23] E. Novak and H. Woźniakowski, *Tractability of multivariate problems. Volume II: Standard informations for functionals*, EMS, Zurich, 2010.

[24] D. Nuyens and R. Cools, *Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces*, Math. Comp., 75 (2006) pp. 903–920.

[25] D. Nuyens and R. Cools, *Fast component-by-component construction, a reprise for different kernels*, In: Monte Carlo and Quasi-Monte Carlo Methods 2004, H. Niederreiter and D.

Talay (Eds.), Springer, Berlin, 2006, pp. 373–387.

[26] A. B. Owen, *Randomly permuted (t, m, s)-nets and (t, s)-sequences*, In: Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing. H. Niederreiter and J.-S. Shiue (Eds.), Springer, New York, 1995, pp. 299–317.

[27] A. B. Owen, *Monte Carlo variance of scrambled net quadrature*, SIAM. J. Numer. Anal., 34 (1997) pp. 1884–1910.

[28] A. B. Owen, *Scrambled net variance for integrals of smooth functions*, Ann. Statist., 25 (1997) pp. 1541–1562.

[29] S. H. Paskov and J. F. Traub, *Faster valuation of financial derivatives*, J. Portfolio Manage., 22 (1995) pp. 113–120.

[30] P. D. Proinov, *On the $\mathcal{L}_2$ discrepancy of some infinite sequences*, Serdica, 11 (1985) pp. 3–12.

[31] K. F. Roth, *On irregularities of distribution*, Mathematika, 1 (1954) pp. 73–79.

[32] I. H. Sloan and S. Joe, *Lattice Methods for Multiple Integration*, Oxford University Press, Oxford, 1994.

[33] I. H. Sloan and A. V. Reztsov, *Component-by-component construction of good lattice rules*, Math. Comp., 71 (2002) pp. 263–273.

[34] I. H. Sloan and H. Woźniakowski, *When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?*, J. Complexity, 14 (1998) pp. 1–33.

[35] I. M. Sobol', *Distribution of points in a cube and approximate evaluation of integrals* (in Russian), Zh. Vycisl. Mat. i Mat. Fiz., 7 (1967) pp. 784–802.

[36] S. Tezuka and H. Faure, *I-binomial scrambling of digital nets and sequences*, J. Complexity, 19 (2003) pp. 744–757.

[37] J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math., 45 (1923) pp. 5–24.

[38] H. Woźniakowski, *Average case complexity of multivariate integration*, Bull. Amer. Math. Soc. New Series, 24 (1991) pp. 185–194.

[39] S. K. Zaremba, *Some applications of multidimensional integration by parts*, Ann. Polon. Math., 21 (1968) pp. 85–96.